A classification scheme for non-rotating inhomogeneous cosmologies

1979 J. Phys. A: Math. Gen. 12 2015

(http://iopscience.iop.org/0305-4470/12/11/014)

J. Phys. A: Math. Gen., Vol. 12, No. 11, 1979. Printed in Great Britain

A classification scheme for non-rotating inhomogeneous cosmologies

J Wainwright

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Received 10 January 1979

Abstract. This paper describes a classification scheme for irrotational cosmological models which is not based on the existence of a group of local isometries and hence is suitable for studying inhomogeneous cosmologies. The scheme is based on the algebraic structure of three trace-free symmetric two-index tensors which are defined in such models, namely the shear tensor of the fluid congruence, assumed irrotational, and the trace-free Ricci and Cotton–York tensors associated with the hypersurfaces orthogonal to the fluid. The restrictions that are imposed on these tensors by the existence of various groups of local isometries are derived, thereby relating the present approach to the usual classifications involving Killing vectors. These results lead to the conjecture that the algebraic structure of the Cotton–York tensor (whose vanishing is a necessary and sufficient condition for the hypersurfaces to be conformally flat) is related to the nature of the gravitational waves that might be present.

1. Introduction

The aim of this paper is to provide a mathematical framework for studying local properties of space–times which satisfy the Einstein field equations with an irrotational (perfect) fluid, and possibly an electromagnetic field as source. Our main interest, however, is in applying this to the study of spatially inhomogeneous cosmological models of this type. By 'spatially inhomogeneous' we mean that the space-like hypersurfaces orthogonal to the fluid flow are not the orbits of a local group of isometries. The structure associated with this problem is a space–time, on which is defined a unit future-pointing time-like vector field u which is irrotational and hence determines locally a family of space-like hypersurfaces.

The problem can thus be regarded as one of classifying three-dimensional Riemannian geometries (i.e. the intrinsic geometry of the hypersurfaces) and of classifying normal (i.e. irrotational) time-like congruences. The properties of the normal time-like congruence of course relate to the way in which the hypersurfaces are imbedded in space–time (i.e. their extrinsic geometry). This classification is thus a purely geometrical matter and is independent of any field equations.

A programme of this nature was recently initiated by Collins and Szafron (1979), who presented a classification which is essentially based on the Ricci tensor of the metric induced on the hypersurfaces and on the shear tensor of the normal congruence. They were eventually led to investigating space–times in which the normal congruence is geodesic and the space-like hypersurfaces are conformally flat. Their approach led to a characterisation of the Szekeres (1975) inhomogeneous cosmologies. In order to set up a classification scheme which is sufficiently general to distinguish the various known inhomogeneous solutions, we have been compelled to consider additional quantities.

Firstly consider the normal congruence. The (rate of) shear tensor σ_{ij} (see e.g. Ellis 1971 for standard notation) describes the anisotropy in this congruence. In order to describe its possible inhomogeneity, we found it necessary to consider quantities which are related to the gradients of scalars. Two natural choices are the acceleration vector $\dot{u}^i$ of the congruence (which is related to the spatial pressure gradient when the Einstein field equations with perfect fluid source hold) and the spatial gradient of the expansion scalar $\theta = u^i{}_{;i}$, defined by

$$\chi_i = \theta_{,j} h^j{}_i,$$

where

$$h^j{}_i = \delta^j{}_i + u^j u_i$$

is the projection tensor onto the hypersurfaces. One could also consider the spatial gradients of scalars such as $\sigma_{ij}\sigma^{ij}$, $\dot{u}^i\dot{u}_i$, $\sigma_{ij}\dot{u}^i\dot{u}^j$, but $\dot{u}_i$ and χ_i will be sufficient for our purposes.

Secondly consider the intrinsic geometry of the hypersurfaces. When classifying four-dimensional geometries one makes use of the irreducible parts of the Riemann–Christoffel tensor, i.e. the trace-free Ricci tensor, the Ricci scalar and the Weyl conformal curvature tensor. In a three-dimensional geometry, however, the Weyl tensor is identically zero, and a necessary and sufficient condition for conformal flatness has to involve third derivatives of the metric components. Such conditions were given by Cotton (1899) and Schouten (1921) and are described most conveniently by a rank-two symmetric trace-free tensor† defined by (York 1971)

$$C^{ij} = 2\eta^{irs}(R^j{}_r - \tfrac{1}{4}\delta^j{}_r R)_{;s}. \tag{1.1}$$

Here the indices run from 1 to 3, and all quantities refer to a 3D (positive definite) metric tensor. We will refer to this tensor density as the Cotton–York tensor. (York in fact defined a tensor density.) We stress that this tensor is conformally invariant, in the sense that $\tilde{g}_{ij} = \Omega^2 g_{ij}$ implies that $\Omega\tilde{C}_{ij} = C_{ij}$, and also has the important property that $C_{ij} = 0$ if and only if the 3D geometry is conformally flat. In view of this discussion, it is natural to use the trace-free Ricci tensor, the Ricci scalar and the Cotton–York tensor to classify 3D Riemannian spaces.

We have also found it necessary to consider the spatial gradient of some scalar associated with the three-geometry in order to describe the presence of inhomogeneities adequately. Of all the curvature invariants of the 3D-geometry, the most natural choice, in our opinion, is the Ricci scalar R^*, since R^* is related algebraically to the energy density and kinematic quantities when the Einstein field equations are satisfied, with a general matter field as source. The specific relationship, assuming that the time-like eigenvector of the stress–energy tensor is irrotational and that the cosmological constant is zero, is

$$R^* = 2(\sigma^2 - \tfrac{1}{3}\theta^2 + \mu), \tag{1.2}$$

where μ is the energy density and $\sigma^2 = \sigma_{ij}\sigma^{ij}$ (Ellis 1971). Thus we consider the spatial gradient of the spatial Ricci scalar

$$\Sigma_i = R^*_{,j} h^j{}_i. \tag{1.3}$$

† Relative to coordinate transformations with positive Jacobian.

In § 2 we describe the classification schemes for normal time-like congruences and 3D Riemannian geometries. Sections 3–5 investigate the relationship between various previously studied classes of cosmological models and the present classification scheme. In particular we study to what extent the existence of groups of local isometries in space–time restricts the normal time-like congruence and the intrinsic geometry. In § 3 we consider two-parameter groups with space-like orbits; in § 4 we consider three-parameter groups with 3D space-like orbits, while § 5 is concerned with groups which act multiply transitively, i.e. such that the dimension of the orbits is less than the dimension of the group. In each case it is assumed that either the family of orbits coincides with the given family $\mathcal{F}$ of space-like hypersurfaces, or that each orbit forms a subset of a hypersurface in $\mathcal{F}$. In § 6 we discuss the results.

The statements of the theorems presuppose a knowledge of standard terminology associated with local groups of isometries (see e.g. Eisenhart 1964) and (in § 4) of some of the more specialised terminology associated with spatially homogeneous space–times (see Ellis and MacCallum 1969). The presentation of the proofs of the theorems presupposes a working knowledge of the orthonormal tetrad formalism as presented by MacCallum (1973). The specific formulae that are required are listed in the Appendix.

2. The classification scheme

We use an orthonormal frame $\{e_a\}$, $a = 0, 1, 2, 3$, with $e_0 = u$, the unit tangent vector of the normal time-like congruence. The e_α, $\alpha = 1, 2, 3$, form an orthonormal frame on each space-like hypersurface.

2.1. Classification of normal time-like congruences

We denote the spatial frame components of the shear tensor, acceleration vector and expansion gradient respectively by $\sigma_{\alpha\beta}$, $\dot{u}_\alpha$ and χ_α, with $\alpha, \beta = 1, 2, 3$. Since $\sigma_{ij}u^j = 0$, $\dot{u}_i u^i = 0$, $\chi_i u^i = 0$, all other frame components of these tensors are identically zero. The spatial orthonormal frame relative to which the 3×3 trace-free symmetric matrix $\sigma_{\alpha\beta}$ is diagonal is called a shear eigenframe. The classification is based on the extent to which the vectors $\dot{u}_i$ and χ_i are aligned with each other and with the shear eigenframe.

Class A: $\sigma_{\alpha\beta} \neq 0$; $\dot{u}_\alpha$ is not a shear eigenvector and is not equal to zero.

- A_1: $\chi_{[\alpha}\dot{u}_{\beta]} \neq 0$; χ_α is not a shear eigenvector.
- A_2: $\chi_{[\alpha}\dot{u}_{\beta]} \neq 0$; χ_α is a shear eigenvector.
- A_3: $\chi_{[\alpha}\dot{u}_{\beta]} = 0$; $\chi_\alpha \neq 0$.
- A_4: $\chi_\alpha = 0$.

Class B: $\sigma_{\alpha\beta} \neq 0$; $\dot{u}_\alpha$ is a shear eigenvector.

- B_1: $\chi_{[\alpha}\dot{u}_{\beta]} \neq 0$; χ_α is not a shear eigenvector.
- B_2: $\chi_{[\alpha}\dot{u}_{\beta]} \neq 0$; χ_α is a shear eigenvector.
- B_3: $\chi_{[\alpha}\dot{u}_{\beta]} = 0$; $\chi_\alpha \neq 0$.
- B_4: $\chi_\alpha = 0$.

Class C: $\sigma_{\alpha\beta} \neq 0$; $\dot{u}_\alpha = 0$.

- C_1: $\chi_\alpha \neq 0$; χ_a is not a shear eigenvector.
- C_2: $\chi_\alpha \neq 0$; χ_α is a shear eigenvector.
- C_3: $\chi_\alpha = 0$.

Class D: $\sigma_{\alpha\beta}=0$; $\dot{u}_\alpha \neq 0$.

D$_1$: $\chi_{[\alpha}\dot{u}_{\beta]} \neq 0$.

D$_2$: $\chi_{[\alpha}\dot{u}_{\beta]}=0$; $\chi_\alpha \neq 0$.

D$_3$: $\chi_\alpha = 0$.

Class E: $\sigma_{\alpha\beta}=0$; $\dot{u}_\alpha = 0$.

E$_1$: $\chi_\alpha \neq 0$.

E$_2$: $\chi_\alpha = 0$.

Classes A–C can be subclassified according to whether $\sigma_{\alpha\beta}$ has a repeated eigenvalue.

2.2. *Classification of 3D Riemannian spaces*

Let $R^*_{\alpha\beta}$, $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ denote the frame components of the Ricci, trace-free Ricci and Cotton–York tensors of the induced metric on the hypersurfaces. Let Σ_α denote the frame components of the spatial gradient of R^*, the associated Ricci scalar (see equation (1.3)). A spatial orthonormal frame relative to which $R^*_{\alpha\beta}$ (or $S^*_{\alpha\beta}$) is diagonal will be called a Ricci eigenframe, and one relative to which $C^*_{\alpha\beta}$ is diagonal will be called a Cotton–York eigenframe. The classification is based on the extent to which these eigenframes are aligned.

Class I: $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ have no eigenvectors in common.

Class II: $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ have one common eigenvector.

Class III: $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ have a common eigenframe.

Class IV: $C^*_{\alpha\beta}=\lambda S^*_{\alpha\beta}$, $\lambda \neq 0$.

Class V: $C^*_{\alpha\beta}=0$.

Class VI: $R^*_{\alpha\beta}=0$.

Classes I–V can, where appropriate, be subclassified according to whether or not:

(1) $S^*_{\alpha\beta}$ and/or $C^*_{\alpha\beta}$ admit repeated eigenvalues;

(2) $S^*_{\alpha\beta}$ and/or $C^*_{\alpha\beta}$ admit a zero eigenvalue;

(3) $R^*=0$;

(4) the spatial gradient of R^* is an eigenvector of $S^*_{\alpha\beta}$ and/or $C^*_{\alpha\beta}$.

3. Two-parameter groups of local isometries

We consider space–times which admit a two-parameter group G_2 of local isometries. The group orbits are assumed to be orthogonal to the given time-like normal congruence, and hence must be space-like two-surfaces. There are exactly two canonically distinct types of G_2's: Abelian and non-Abelian. In the former case the Killing vectors ξ, η of the group have zero Lie bracket, $[\xi, \eta]=0$, while in the latter case the Killing vectors may be chosen so that $[\xi, \eta]=\xi$. Both cases can be subclassified according to whether or not the group acts orthogonally transitively, i.e. whether or not the two-spaces orthogonal to the group orbits generate two-surfaces. A necessary and sufficient condition for this is that

$$\xi_{[i,j}\,\xi_k\eta_{l]}=0, \qquad \eta_{[i,j}\eta_k\xi_{l]}=0$$

(see e.g. Carter 1973). Alternatively, if we specialise the orthonormal frame of § 2 so that e_2, e_3 are tangent to the group orbits (as will be done in the remainder of this section), the group will act orthogonally transitively if and only if e_0, e_1 generate two-surfaces, i.e. if and only if

$$[e_0, e_1] = fe_0 + ge_1 \tag{3.1}$$

for some functions, f, g. Finally there is the possibility that the group has Killing vectors which are hypersurface-orthogonal. Note that this necessarily implies that the group acts orthogonally transitively. In the Abelian case this implies that there exist coordinates relative to which the line element is diagonal.

The last-mentioned situation is the one which has been most studied from the point of view of inhomogeneous cosmologies (e.g. Liang 1976), and even here it is difficult to find exact solutions with a perfect fluid source. Indeed only in the special case of a perfect fluid with equation of state $p = \mu$ has much progress been made in generating exact solutions. In this case it has also proved possible to include Abelian groups which are only assumed to act orthogonally transitively (Wainwright *et al* 1979). Little or no progress has been made in more general situations (e.g. orthogonal transitivity not assumed, or a non-Abelian group).

The following theorem relates space–times admitting a G_2 to the classification scheme of § 2. We stress that it is assumed that the group orbits are contained in the given family of space-like hypersurfaces.

Theorem 3.1. Suppose that a G_2 of local isometries acts on space–time, with its orbits contained in a given family of space-like hypersurfaces. Let v be a unit vector tangent to the hypersurfaces and orthogonal to the group orbits. Then:

(1) the acceleration vector and the spatial gradients of θ and R^* are parallel to v, but v is a shear eigenvector if and only if the group acts orthogonally transitively;
(2) if the group is Abelian, v is an eigenvector of both $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$;
(3) if the group is not Abelian, and v is an eigenvector of $S^*_{\alpha\beta}$, the Cotton–York tensor is zero.

Remark. We see that the various spatial gradients always lie in the preferred direction v, but that v is not in general an eigenvector of $\sigma_{\alpha\beta}$, $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$. However, we have:

Corollary. If $C^*_{\alpha\beta} \neq 0$, then v is a common eigenvector of $\sigma_{\alpha\beta}$, $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ if and only if the group is Abelian and acts orthogonally transitively.

Proof of theorem 3.1. We have chosen the frame $\{e_a\}$ so that $e_0 = u$ and e_2, e_3 are tangent to the group orbits. This means that e_2, e_3 are linear combinations of the Killing vectors:

$$e_2 = A\xi + B\eta, \qquad e_3 = C\xi + D\eta. \tag{3.2}$$

We are free to perform a tetrad rotation of the form

$$e_2' = \cos\psi e_2 + \sin\psi e_3, \qquad e_3' = -\sin\psi e_2 + \cos\psi e_3. \tag{3.3}$$

Our immediate aim is to show that we can use (3.3) to obtain a frame which satisfies

$$[e_a, \xi] = 0, \qquad [e_a, \eta] = 0, \qquad a = 0, 1, 2, 3. \tag{3.4}$$

Firstly (3.4) holds with $a = 0$ on account of the lemma at the end of the Appendix. Secondly, on account of (3.2) and (3.4) with $a = 0$, the commutators $[e_0, e_A]$, $A = 2, 3$, simplify to

$$[e_0, e_A] = \gamma_0{}^B{}_A e_B, \qquad A = 2, 3, \tag{3.5}$$

with summation over $B = 2, 3$. In addition, since e_2, e_3 are tangent to two-surfaces, we have

$$[e_2, e_3] = \gamma_{23}^{\ \ B} e_B, \tag{3.6}$$

with summation over $B = 2, 3$. The form of (3.5) and (3.6) implies that e_0, e_2, e_3 generate hypersurfaces and hence that e_1 is hypersurface-orthogonal. Thus the lemma at the end of the Appendix implies (3.4) with $a = 1$, which in turn yields

$$[e_1, e_A] = \gamma_{1A}^{\ \ B} e_B, \qquad A = 2, 3. \tag{3.7}$$

At this stage it is easy to show that†

$$[e_2, \xi_\Omega] = F_\Omega e_3, \qquad [e_3, \xi_\Omega] = -F_\Omega e_2$$

for some functions F_Ω, with $\Omega = 0, 1$ and $\xi_0 = \xi$, $\xi_1 = \eta$. A short calculation now shows that we can obtain (3.4) with $a = 2, 3$ by using (3.3), provided that we can choose ψ to satisfy the first-order partial differential equations

$$\xi_\Omega(\psi) = F_\Omega, \qquad \Omega = 0, 1.$$

The integrability conditions for this system are

$$\xi_\Omega(F_\Lambda) - \xi_\Lambda(F_\Omega) = C^{\Gamma}_{\ \Omega\Lambda} F_\Gamma,$$

where the C's are the structure constants of the group. That these conditions are identically satisfied is a consequence of the Jacobi identity (A2) applied to ξ_Ω, ξ_Λ and e_2. Note that one can still use the tetrad freedom (3.3) with ψ constant on the group orbits. Having established (3.4), the Jacobi identities (A2) applied to ξ_Ω, e_a, e_b imply that the objects of anholonomity $\gamma_{ab}^{\ \ c}$ are constant on the group orbits, i.e.

$$\partial_2 \gamma_{ab}^{\ \ c} = 0 = \partial_3 \gamma_{ab}^{\ \ c},$$

where

$$\partial_a f = e_a(f) = e_a^{\ i} f_{,i}.$$

We can now proceed with the main part of the proof. On comparing (3.5) and (A1) and recalling that e_0 is irrotational by assumption, we find that

$$\dot{u}_2 = \dot{u}_3 = 0, \qquad \sigma_{12} + \Omega_3 = 0, \qquad \sigma_{13} - \Omega_2 = 0, \tag{3.8}$$

and hence that

$$[e_0, e_1] = \dot{u}_1 e_0 - \theta_1 e_1 - 2\sigma_{12} e_2 - 2\sigma_{13} e_3.$$

It follows from this equation and the discussion preceding (3.1) that $v(\equiv e_1)$ is a shear eigenvector, i.e. $\sigma_{12} = \sigma_{13} = 0$, if and only if the group acts orthogonally transitively. The remainder of part (1) of the theorem follows from (3.8) and the fact that for any invariantly defined scalar f the spatial gradient of f is orthogonal to ξ, η and hence parallel to e_1.

On comparing (3.6), (3.7) with (A1), we obtain

$$n_{11} = 0, \qquad n_{13} - a_2 = 0, \qquad n_{12} + a_3 = 0$$

and hence

$$[e_2, e_3] = -2a_3 e_2 + 2a_2 e_3.$$

† This approach was suggested by C B Collins (private communication).

At this stage we can note that the group is Abelian, i.e. $[\xi, \eta]=0$, if and only if $[e_2, e_3]=0$, which is true if and only if $a_2=a_3=0$.

Equations (A4) now yield, using (A3) and the simplifications obtained so far, the formulae

$$R^*_{12}=-2a_3(n_{22}-n_{33})+4a_2n_{23}, \qquad R^*_{13}=-2a_2(n_{22}-n_{33})-4a_3n_{23}.$$

Thus if the group is Abelian (i.e. $a_2=a_3=0$), e_1 is an eigenvector of $R^*_{\alpha\beta}$. In general the frame in use is not a Ricci eigenframe. However, since the $R^*_{\alpha\beta}$ are constant on the group orbits, we can use (3.3), with $\partial_2\psi=\partial_3\psi=0$, to achieve $R^*_{23}=0$, i.e. to introduce a Ricci eigenframe. Then we can use (A5) to calculate $C^*_{\alpha\beta}$, and in the Abelian case we find that $C^*_{12}=C^*_{13}=0$, i.e. e_1 is an eigenvector of $C^*_{\alpha\beta}$. This completes the proof of part (2).

To prove part (3), we note that e_1 is an eigenvector of $R^*_{\alpha\beta}$ if and only if $a_2=a_3=0$ or $n_{22}=n_{33}$, $n_{23}=0$. In the latter case (A4) implies that $R^*_{23}=0$, i.e. the frame in use is a Ricci eigenframe. This permits us to use (A5) to conclude that $C^*_{\alpha\beta}=0$, completing the proof of part (3) of the theorem. □

4. Spatially homogeneous space–times

Suppose that space–time is spatially homogeneous, i.e. it admits a G_3 of local isometries whose orbits are space-like hypersurfaces. We can introduce an orthonormal frame $\{e_a\}$ with e_0 chosen to be the unit normal of the group orbits and such that

$$[e_a, \xi_A]=0,$$

where the ξ_A, $A=1, 2, 3$, are Killing vectors which generate the group (see Ellis and MacCallum (1969) for this and other results which we quote). Relative to such a frame the objects of anholonomity $\gamma_{ab}{}^c$ are constant on the group orbits. Their spatial components $\gamma_{\alpha\beta}{}^\mu$ can be decomposed algebraically into quantities $n_{\alpha\beta}$ and a_α, which satisfy

$$n_{\alpha\beta}=n_{\beta\alpha}, \qquad n^{\beta}_{\alpha}a_\beta=0$$

with summation over the repeated β. Under rotations of the spatial frame, which are restricted to be constant on the orbits, $n_{\alpha\beta}$ and a_β transform as components of a rank-two tensor and vector respectively. One can thus choose the e_α to be an eigenframe of $n_{\alpha\beta}$, and in addition, if $a_\alpha\neq 0$, one can without loss of generality choose the eigenframe so that a_α has a component only in the e_1 direction:

$$(n_{\alpha\beta})=\mathrm{diag}(n_1, n_2, n_3), \qquad (a_\alpha)=(a, 0, 0).$$

We refer to a frame with these properties as a canonical frame in a spatially homogeneous space–time. Space–times with $a_\alpha=0$ are said to be of class A and those with $a_\alpha\neq 0$ are said to be of class B in the Ellis and MacCallum classification.

Theorem 4.1. In any spatially homogeneous space–time, the vector e_1 of a canonical frame is an eigenvector of both the spatial Ricci and Cotton–York tensors of the group orbits. If $S^*_{\alpha\beta}\neq 0$, then a canonical frame is a common eigenframe of $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ if and only if the space–time is of class A ($a_\alpha=0$). The acceleration vector and spatial gradients of R^* and θ are zero.

Proof. Relative to a canonical frame, equations (A4) yield

$$R^*_{12} = R^*_{13} = 0, \qquad R^*_{23} = a_1(n_{22} - n_{33}). \tag{4.1}$$

Thus e_1 is a Ricci eigenvector, and in class A space–times ($a_1 = 0$) the canonical frame is a Ricci eigenframe. Further use of (A4) shows that in class B space–times ($a_1 \neq 0$) a canonical frame is a Ricci eigenframe if and only if $S^*_{\alpha\beta} = 0$.

In this situation one can perform a change of frame of the form (3.3), with ψ constant on the group orbits, to achieve $R^*_{23} = 0$, i.e. so that the new frame is a Ricci eigenframe, but not necessarily a canonical frame. Then in general $n_{23} \neq 0$. Equations (A5) can now be used to conclude that $C^*_{12} = C^*_{13} = 0$, i.e. e_1 is a Cotton–York eigenvector. In addition in class A space–times relative to a canonical frame ($a = 0$, $n_{23} = 0$) it follows that $C^*_{23} = 0$, i.e. a canonical frame is a Cotton–York eigenframe. □

Remarks. (1) A canonical frame is not in general a shear eigenframe, but it is, for example, if the space–time is of class A and the Einstein field equations with a perfect fluid source hold (Ellis and MacCallum 1969).

(2) The following theorems cover various special cases (cf Spero and Szafron (1978) for a detailed analysis of the case $C^*_{\alpha\beta} = 0$).

Theorem 4.2. In class A space–times, if $n_{\alpha\beta}$ has a repeated eigenvalue, then $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ have a repeated eigenvalue and $C^*_{\alpha\beta} = \lambda S^*_{\alpha\beta}$. If $S^*_{\alpha\beta} \neq 0$, then $C^*_{\alpha\beta} \neq 0$ and the following Bianchi–Behr group types are possible: II, VIII and IX.

Proof. Since $n_{\alpha\beta}$ has a repeated eigenvalue and the space–time is of class A, we may assume without loss of generality that $n_{22} = n_{33}$. Using equations (A4) and (A5) we quickly find that

$$C^*_{\alpha\beta} = 3n_{11}S^*_{\alpha\beta},$$

with

$$S^*_{11} = -2S^*_{22} = -2S^*_{33} = \tfrac{2}{3}n_{11}(n_{11} - n_{22}),$$

relative to a canonical frame. The theorem now follows using the Bianchi–Behr classification (see Ellis and MacCallum 1969). □

Theorem 4.3. In class B space–times with $S^*_{\alpha\beta} \neq 0$, $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ have a common eigenframe if and only if $a_1^2 + n_2 n_3 = 0$ in the canonical frame, i.e. if and only if the group is of Bianchi–Behr type $\mathrm{VI}_{h=-1}$. In this situation $S^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ have a repeated eigenvalue and $C^*_{\alpha\beta} = \lambda S^*_{\alpha\beta}$, with $\lambda = 0$ if and only if $n_2 = -n_3$ in the canonical frame (i.e. $n_\alpha{}^\alpha = 0$).

Proof. Since $S^*_{\alpha\beta} \neq 0$, it follows from the proof of theorem 4.1 that a canonical frame is not a Ricci eigenframe. We thus transform to a Ricci eigenframe as in the proof of theorem 4.1. The requirement that $R^*_{23} = 0$ is, using equations (A4),

$$a_1(n_{22} - n_{33}) + n_{23}(n_{22} + n_{33}) = 0, \tag{4.2}$$

where now

$$n_{23} \neq 0. \tag{4.3}$$

In addition, equations (A4) in a canonical frame imply that $n_{22} \neq n_{33}$, which in a Ricci eigenframe becomes

$$(n_{22}-n_{33})^2+4n_{23}^2 \neq 0, \tag{4.4}$$

since this quantity is invariant under (3.3) (Collins and Szafron 1979). At this stage the non-zero components of $R^*_{\alpha\beta}$ in the Ricci eigenframe are

$$\begin{aligned} R^*_{11} &= -\tfrac{1}{2}(n_{22}-n_{33})^2-2n_{23}^2-2a_1^2, \\ R^*_{22} &= \tfrac{1}{2}(n_{22}^2-n_{33}^2)-2a_1(a_1+n_{23}), \\ R^*_{33} &= -\tfrac{1}{2}(n_{22}^2-n_{33}^2)-2a_1(a_1-n_{23}). \end{aligned}$$

By using these equations together with equations (4.2) and (A5) we find that the only non-zero off-diagonal term of $C^*_{\alpha\beta}$ is

$$C^*_{23} = -4n_{23}(a_1^2+n_{22}n_{33}-n_{23}^2).$$

Thus on account of (4.3), $C^*_{23}=0$ if and only if

$$a_1^2+n_{22}n_{33}-n_{23}^2=0, \tag{4.5}$$

i.e. if and only if

$$a_1^2+n_2n_3=0, \tag{4.6}$$

where n_2, n_3 are the two non-zero eigenvalues of $n_{\alpha\beta}$.

Note that, on account of (4.2) and (4.4), $R^*_{22}-R^*_{33}\neq 0$, so that C^*_{23} cannot be transformed to zero. In other words, if $S^*_{\alpha\beta}\neq 0$, a Ricci eigenframe is a Cotton–York eigenframe if and only if relation (4.5) holds, i.e. if and only if the group is of type VI_h, with $h=-1$ (Ellis and MacCallum 1969).

An analysis of (4.2) and (4.5) subject to (4.4) implies that $n_{22}n_{33}(n_{22}-n_{33})=0$. If $n_{22}=n_{33}$, which on account of (4.2) and (4.3) implies $n_{22}=0=n_{33}$ (i.e. the eigenvalues satisfy $n_2=-n_3$), equations (A5) imply that $C^*_{\alpha\beta}=0$. If $n_{22}=0$, $n_{33}\neq 0$, it follows that $R^*_{11}=R^*_{22}$, and subsequently, using equations (A5), that

$$C^*_{11}=C^*_{22}=n_{33}(R^*_{11}-R^*_{33}).$$

Hence $C^*_{\alpha\beta}=\lambda S^*_{\alpha\beta}$. The remaining case is similar. □

Corollary. If $C^*_{\alpha\beta}\neq\lambda S^*_{\alpha\beta}\neq 0$ in a spatially homogeneous space–time, then $C^*_{\alpha\beta}$ and $S^*_{\alpha\beta}$ have a common eigenframe if and only if the space–time is of class A.

In other words, apart from various degenerate cases, class A and class B spatially homogeneous space–times are distinguished by whether or not $C^*_{\alpha\beta}$ and $S^*_{\alpha\beta}$ have a common eigenframe.

5. Multiply transitive groups of local isometries

We first consider the case of a three-parameter group with 2D orbits which are contained in the preferred hypersurfaces. This means that the space–time admits a one-parameter local isotropy subgroup which acts, at each point, in the 2D subspace tangent to the group orbits. This subgroup leaves invariant the metric, and, by assumption, the normal vector field u. Thus the space–time, the vector field u and the geometry of the hypersurfaces are locally rotationally symmetric (abbreviated LRS; see Ellis 1967). We have the following:

Theorem 5.1. Suppose that space–time admits a G_3 of local isometries with 2D orbits contained in the given hypersurfaces. Let v be a unit vector tangent to the hypersurfaces and orthogonal to the group orbits. Then v is an eigenvector of the shear tensor and spatial Ricci tensor, and the spatial Cotton–York tensor vanishes. The spatial gradients of R^* and θ and the acceleration vector of the normal congruence are parallel to v. In addition the spatial Ricci and shear tensors each have a repeated eigenvalue, and the associated 2D eigenspace is orthogonal to v.

Remark. The assertions that comprise this and the next theorem, except possibly for those pertaining to the Cotton–York tensor, will be fairly obvious to anyone familiar with local rotational symmetry. For this reason, the proofs are only given in outline at the end of the section.

We now consider the case of a four-parameter group with 3D orbits which are assumed to be the given hypersurfaces. We can assume that the group does not admit a three-parameter subgroup with 2D orbits, since this is included in theorem 5.1. The space–time still admits a one-parameter local isotropy group which induces a group of rotations in a 2D subspace of the tangent space at each point. Let v be a unit vector which is orthogonal to this 2D subspace at each point.

Theorem 5.2. Suppose that space–time admits a G_4 of local isometries with 3D space-like orbits, and there exists no three-parameter subgroup with 2D orbits. Then the Cotton–York tensor and the trace-free Ricci tensor of the metric induced on the group orbits are related according to

$$C^*_{\alpha\beta} = \lambda S^*_{\alpha\beta},$$

where $\lambda \neq 0$ is constant on the orbits. In addition $S^*_{\alpha\beta}$ (and hence $C^*_{\alpha\beta}$) and the shear tensor of the normal congruence have a repeated eigenvalue, with the associated 2D eigenspace orthogonal to the vector v. The acceleration vector and the spatial gradients of R^* and θ are zero.

Proof of theorems 5.1 and 5.2. Choose an orthonormal frame with $e_o = u$, $e_1 = v$. The discussion preceding the theorems indicates that in each case the space–time, the vector field u and the geometry of the hypersurfaces are LRS relative to the two-space spanned by e_2 and e_3, i.e. at each point all directions in this two-space are equivalent. Hence each vector in this two-space must be an eigenvector of the shear tensor and the spatial Ricci and Cotton–York tensor (unless any of these tensors is zero). Thus the chosen spatial orthonormal frame is an eigenframe of each of these tensors. In addition any geometrically defined vector orthogonal to u, e.g. u and the spatial gradients of θ and R^*, must be orthogonal to the two-space spanned by e_2, e_3, and hence parallel to v, unless they vanish, which is the case in theorem 5.2. Theorems 5.1 and 5.2 are also distinguished by the fact that in the first one e_2 and e_3 are tangent to the 2D group orbits, so that $[e_2, e_3]$ is a linear combination of e_2 and e_3. Equations (A1) then imply that $n_{11} = 0$, i.e. $n_{11} = 0$ in theorem 5.1 and $n_{11} \neq 0$ in theorem 5.2. To complete the proof we need to calculate $C^*_{\alpha\beta}$. In an LRS space–time with a frame as chosen we have

$$\partial_2 \gamma^c{}_{ab} = \partial_3 \gamma^c{}_{ab} = 0$$

and

$$n_{22} = n_{33}, \qquad n_{23} = 0, \qquad n_{12} = -a_3, \qquad n_{13} = a_2$$

(see Ellis and MacCallum 1969, p 126), and from the above remarks

$$R^*_{22} = R^*_{33}.$$

Using these results, equations (A5) quickly yield

$$C^*_{11} = -2C^*_{22} = -2C^*_{33} = 2n_{11}(R^*_{11} - R^*_{22}).$$

(Recall that we have already established that the frame in use is an eigenframe of $C^*_{\alpha\beta}$.) Since $n_{11} = 0$ in theorem 5.1 and $n_{11} \neq 0$ in theorem 5.2, this completes the proof. □

6. Conclusions

The results presented describe the restrictions imposed on the intrinsic and extrinsic geometry of a given family of space-like hypersurfaces by the existence of various groups of isometries, and enable us to relate these classes of space–times to the classification scheme in § 2. We briefly summarise the relationship, which shows that the classification scheme provides a clear distinction between the various isometry groups.

(1) *Two-parameter group with space-like orbits.* In general the extrinsic geometry is A_3 and the intrinsic geometry is I. If the group acts orthogonally transitively, the extrinsic geometry is in general B_3. If the group is Abelian, the intrinsic geometry is in general II.

It is interesting that orthogonal transitivity only affects the extrinsic behaviour, while commutativity of the group only affects the intrinsic behaviour.

(2) *Three-parameter group with 3D space-like orbits.* In general the extrinsic geometry is C_3. The Ellis–MacCallum A space–times have intrinsic geometry of class III, while the B space–times have intrinsic geometry of class II in general.

(3) *Three-parameter group with 2D space-like orbits.* The extrinsic and intrinsic geometries are respectively B_3 and V in general.

(4) *Four-parameter group with 3D space-like orbits.* The extrinsic and intrinsic geometries are respectively C_3 and IV in general.

The results of this paper are independent of any field equations. The role of the Einstein field equations can be seen by inspection of the tetrad form of the field equations as given by MacCallum (1973, equations (82)–(84)). The (0, 0) and $(0, \alpha)$ equations relate extrinsic quantities and their derivatives to source terms. On the other hand the (α, β) equations express the intrinsic Ricci components $R^*_{\alpha\beta}$ in terms of extrinsic quantities, their derivatives and source terms. Thus the Einstein field equations relate the intrinsic geometry to the extrinsic geometry, the strength of this relationship depending on the complexity of the source terms, which are derived from the energy–momentum tensor. For example, if the Einstein field equations with perfect fluid source hold in a class A spatially homogeneous space–time, then a Fermi-propagated shear eigenframe is necessarily an eigenframe of the spatial Ricci and Cotton–York tensors, as follows from theorem 4.1 and Ellis and MacCallum (1969, p 118).

In a subsequent paper we shall analyse the various classes of known inhomogeneous cosmologies from the point of view of the classification scheme presented in this paper. Some of these exact solutions—such as the Szekeres (1975) solutions (see also Szafron 1977, Szafron and Wainwright 1977), the type-N perfect solutions of Oleson (1971), and a class of algebraically special solutions (Wainwright 1974)—do not admit any Killing vectors in general (see Bonnor *et al* (1977) in connection with the Szekeres solutions). Apart from the Szekeres solutions, which have zero acceleration and zero Cotton–York tensor (Berger *et al* 1977, Szafron and Collins 1979), we will show that these solutions in general satisfy no restrictions as regards the present classification scheme.

Finally we discuss a possible physical significance of the intrinsic classification. Berger *et al* (1977) have suggested that there may be some relation between the conformal curvature of three-slices in space–time and the presence of gravitational waves. Indeed it was this remark that first directed our attention to the Cotton–York tensor. This conjecture appears to be borne out in the case of the Szekeres solutions, since the Cotton–York tensor vanishes, and the work of Bonnor (1977) implies that they are indeed non-radiative. Furthermore, Lukash (1976) has shown that certain spatially homogeneous perfect fluid solutions can be interpreted as containing gravitational waves. His analysis leads to the conclusion that the Bianchi–Behr type VII_0 solutions, which are of class A, contain standing waves only, while the type VII_h solutions, which are of class B and hence have a more complicated Cotton–York tensor (cf theorem 4.1), can admit travelling waves. In both cases the Cotton–York tensor is non-zero. Whether or not the algebraic structure of the Cotton–York tensor is in general related to the nature of the gravitational waves requires further investigation, however.

Appendix

This Appendix contains formulae which relate to the orthonormal tetrad formalism as presented by MacCallum (1973). All the formulae, with the exception of those relating to the Cotton–York tensor, which we believe are new, are taken from MacCallum (1973). The commutators of the frame vectors e_a, $a = 0, 1, 2, 3$, which have the general form

$$[e_a, e_b] = \gamma^c{}_{ab} e_c$$

are written out explicitly as

$$\begin{aligned}
[e_0, e_1] &= \dot{u}^1 e_0 - \theta_1 e_1 - (\sigma_{12} - \omega_3 - \Omega_3) e_2 - (\sigma_{13} + \omega_2 + \Omega_2) e_3,\\
[e_0, e_2] &= \dot{u}^2 e_0 - (\sigma_{12} + \omega_3 + \Omega_3) e_1 - \theta_2 e_2 - (\sigma_{23} - \omega_1 - \Omega_1) e_3,\\
[e_0, e_3] &= \dot{u}^3 e_0 - (\sigma_{13} - \omega_2 - \Omega_2) e_1 - (\sigma_{23} + \omega_1 + \Omega_1) e_2 - \theta_3 e_3,\\
[e_1, e_2] &= -2\omega_3 e_0 + (n_{13} - a_2) e_1 + (n_{23} + a_1) e_2 + n_{33} e_3,\\
[e_2, e_3] &= -2\omega_1 e_0 + n_{11} e_1 + (n_{12} - a_3) e_2 + (n_{13} + a_2) e_3,\\
[e_3, e_1] &= -2\omega_2 e_0 + (n_{12} + a_3) e_1 + n_{22} e_2 + (n_{23} - a_1) e_3
\end{aligned} \tag{A1}$$

in terms of the kinematical quantities associated with the e_0 congruence and the quantities $n_{\alpha\beta}$, a_α and Ω_α.

We also require the Jacobi identity

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \tag{A2}$$

for arbitrary vector fields X, Y, Z. When applied to the vector fields e_μ, e_σ, e_ν, equation (A2) can be written in the form

$$\partial_\mu n^{\mu\alpha}+\eta^{\alpha\mu\nu}\partial_\mu a_\nu-2\theta^\alpha{}_\beta\omega^\beta-2n^\alpha{}_\beta a^\beta-2\eta^{\alpha\mu\nu}\omega_\mu\Omega_\nu=0.$$

When written out, subject to the requirement that e_0 be irrotational, i.e. $\omega_\alpha=0$, these equations read

$$\begin{aligned}
&\partial_1 n_{11}+\partial_2 n_{12}+\partial_3 n_{31}+\partial_2 a_3-\partial_3 a_2-2(n_{11}a_1+n_{12}a_2+n_{31}a_3)=0,\\
&\partial_2 n_{22}+\partial_3 n_{23}+\partial_1 n_{12}+\partial_3 a_1-\partial_1 a_3-2(n_{22}a_2+n_{23}a_3+n_{12}a_1)=0,\\
&\partial_3 n_{33}+\partial_1 n_{31}+\partial_2 n_{23}+\partial_1 a_2-\partial_2 a_1-2(n_{33}a_3+n_{31}a_1+n_{23}a_2)=0.
\end{aligned} \tag{A3}$$

When e_0 is irrotational, the quantities $R^*_{\alpha\beta}$ of MacCallum (1973) are the frame components of the Ricci tensor induced on the hypersurfaces orthogonal to e_0. The $R^*_{\alpha\beta}$ can be expressed in terms of the $n_{\alpha\beta}$ and a_α according to

$$\begin{aligned}
R^*_{\alpha\beta}=\partial_{(\alpha}a_{\beta)}+\eta_{\mu\nu(\alpha}[\partial^\nu-2a^\nu]n_{\beta)\mu}+2n^\lambda{}_{(\alpha}n_{\beta)\lambda}-nn_{\alpha\beta}\\
-\delta_{\alpha\beta}(2a_\lambda a^\lambda+n^{\lambda\delta}n_{\lambda\delta}-\tfrac{1}{2}n^2-\partial_\lambda a^\lambda).
\end{aligned}$$

For the purposes of this paper it is convenient to write these equations out explicitly. One obtains

$$\begin{aligned}
R^*_{11}&=2\partial_1 a_1+\partial_2(a_2-n_{13})+\partial_3(a_3+n_{12})+2(a_2n_{13}-a_3n_{12})+\tfrac{1}{2}(n_{11}^2-n_{22}^2-n_{33}^2)+n_{22}n_{33}\\
&\quad-2n_{23}^2-2(a_1^2+a_2^2+a_3^2),\\
R^*_{22}&=2\partial_2 a_2+\partial_1(n_{23}+a_1)+\partial_3(a_3-n_{12})+2(a_3n_{12}-a_1n_{23})+\tfrac{1}{2}(n_{22}^2-n_{11}^2-n_{33}^2)+n_{11}n_{33}\\
&\quad-2n_{13}^2-2(a_1^2+a_2^2+a_3^2),\\
R^*_{33}&=2\partial_3 a_3+\partial_1(a_1-n_{23})+\partial_2(a_2+n_{13})+2(a_1n_{23}-a_2n_{13})+\tfrac{1}{2}(n_{33}^2-n_{11}^2-n_{22}^2)+n_{11}n_{22}\\
&\quad-2n_{12}^2-2(a_1^2+a_2^2+a_3^2),\\
R^*_{12}&=\tfrac{1}{2}\partial_1(a_2+n_{13})+\tfrac{1}{2}\partial_2(a_1-n_{23})-\tfrac{1}{2}\partial_3(n_{11}-n_{22})+a_3(n_{11}-n_{22})+a_2n_{23}-a_1n_{13}\\
&\quad+n_{12}(n_{11}+n_{22}-n_{33})+2n_{13}n_{23},\\
R^*_{13}&=\tfrac{1}{2}\partial_1(a_3-n_{12})+\tfrac{1}{2}\partial_3(a_1+n_{23})+\tfrac{1}{2}\partial_2(n_{11}-n_{33})+a_2(n_{33}-n_{11})+a_1n_{12}-a_3n_{23}\\
&\quad+n_{13}(n_{11}-n_{22}+n_{33})+2n_{12}n_{23},\\
R^*_{23}&=\tfrac{1}{2}\partial_2(a_3+n_{12})+\tfrac{1}{2}\partial_3(a_2-n_{13})+\tfrac{1}{2}\partial_1(n_{33}-n_{22})+a_1(n_{22}-n_{33})+a_3n_{13}-a_2n_{12}\\
&\quad+n_{23}(-n_{11}+n_{22}+n_{33})+2n_{12}n_{13}.
\end{aligned} \tag{A4}$$

We now derive a formula for the Cotton–York tensor density as defined by equation (1.1). We prefer to use the associated tensor, i.e. to replace the antisymmetric tensor density ϵ^{ijk} by the tensor η^{ijk} in the definition. The frame components are given by

$$C^{*\alpha\beta}=2\eta^{\alpha\mu\nu}(R^{*\beta}{}_{\mu;\nu}-\tfrac{1}{4}\delta^\beta{}_\mu R^*{}_{;\nu}),$$

where

$$R^{*\beta}{}_{\mu;\nu}=\partial_\nu R^{*\beta}{}_\mu+R^{*\lambda}{}_\mu\Gamma^\beta{}_{\nu\lambda}-R^{*\beta}{}_\lambda\Gamma^\lambda{}_{\nu\mu}.$$

It is in fact more convenient to use a form of $C^{*\alpha\beta}$ in which the symmetry is manifestly obvious, namely

$$C^{*\alpha\beta} = 2\eta^{\mu\nu(\alpha}R^{*\beta)}{}_{\mu;\nu}.$$

On expressing the $\Gamma^{\alpha}{}_{\mu\nu}$ in terms of the $n_{\alpha\beta}$ and a_α, and making standard simplifications, we obtain

$$C^{*\alpha\beta} = 2\eta^{\mu\nu(\alpha}[\partial_\nu - a_\nu]R^{*\beta)}{}_{\mu} + 4n^{\lambda(\alpha}R^{*\beta)}{}_{\lambda} + n_{\alpha\mu\nu}\eta_{\beta\kappa\sigma}R^{*\mu\kappa}n^{\nu\sigma} - R^{*}n_{\alpha\beta} - n_{\mu\nu}R^{*\mu\nu}\delta^{\alpha\beta}.$$

It is convenient to write these equations out explicitly relative to an eigenframe of $R^{*}_{\alpha\beta}$, i.e. subject to $R^{*}_{12} = R^{*}_{13} = R^{*}_{23} = 0$. One obtains

$$\begin{aligned}
C^{*}_{11} &= 2n_{11}R^{*}_{11} + (-n_{11} - n_{22} + n_{33})R^{*}_{22} + (-n_{11} + n_{22} - n_{33})R^{*}_{33},\\
C^{*}_{22} &= (-n_{11} - n_{22} + n_{33})R^{*}_{11} + 2n_{22}R^{*}_{22} + (n_{11} - n_{22} - n_{33})R^{*}_{33},\\
C^{*}_{33} &= (-n_{11} + n_{22} - n_{33})R^{*}_{11} + (n_{11} - n_{22} - n_{33})R^{*}_{22} + 2n_{33}R^{*}_{33},\\
C^{*}_{12} &= -\partial_3(R^{*}_{11} - R^{*}_{22}) + n_{12}(R^{*}_{11} + R^{*}_{22} - 2R^{*}_{33}) + a_3(R^{*}_{11} - R^{*}_{22}),\\
C^{*}_{13} &= -\partial_2(R^{*}_{33} - R^{*}_{11}) + n_{13}(R^{*}_{11} - 2R^{*}_{22} + R^{*}_{33}) + a_2(R^{*}_{33} - R^{*}_{11}),\\
C^{*}_{23} &= -\partial_1(R^{*}_{22} - R^{*}_{33}) + n_{23}(-2R^{*}_{11} + R^{*}_{22} + R^{*}_{33}) + a_1(R^{*}_{22} - R^{*}_{33}).
\end{aligned} \tag{A5}$$

We note that formulae which are equivalent to the vanishing of the expressions in (A5) have been given by Spero and Szafron (1978) and Szafron and Collins (1979).

Finally we require the following:

Lemma. If ξ is a Killing vector field and η is a hypersurface-orthogonal vector field which is orthogonal to ξ, then $[\xi, \eta] = 0$.

The proof is straightforward.

Acknowledgments

I am indebted to C B Collins and B J Marshman for reading and suggesting improvements to the manuscript. I would also like to thank G W Horndeski and, in particular, D Szafron for many helpful discussions. This work was supported in part by a grant from the National Science and Engineering Research Council of Canada.

References

Berger B K, Eardley D M and Oleson D W 1977 *Phys. Rev.* D **16** 3086–8
Bonnor W 1977 *Commun. Math. Phys.* **51** 191–9
Bonnor W B, Sulaiman A H and Tomimura N 1977 *Gen. Rel. Grav.* **8** 549–59.
Carter B 1973 *Black Hole Equilibrium States* in *Black Holes* ed. C DeWitt and B S DeWitt (New York: Gordon and Breach)
Collins C B and Szafron D 1979 *J. Math. Phys.* **20** to appear
Cotton E 1899 *Ann. Fac. Sci. Toulouse (II)* **1** 385–438

Eisenhart L P 1964 *Riemannian Geometry* (Princeton: University Press)
Ellis G F R 1967 *J. Math. Phys.* **8** 1171–94
—— 1971 *Relativistic Cosmology* in *General Relativity and Cosmology, Proc. Int. School of Physics Enrico Fermi Course XLVII, 1969* ed. R K Sachs (London and New York: Academic)
Ellis G F R and MacCallum M A H 1969 *Commun. Math. Phys.* **12** 108–41
Liang E P 1976 *Astrophys. J.* **204** 235–50
Lukash V N 1976 *Nuovo Cim.* **35B** 268–92
MacCallum M A H 1973 *Cosmological Models from a Geometric Point of View* in *Cargese Lectures in Physics 6, Lectures at the International Summer School of Physics, Cargese, Corsica 1971* ed. E Schatzman (New York: Gordon and Breach)
Oleson M 1971 *J. Math. Phys.* **12** 666–72
Schouten J A 1921 *Math. Z.* **11** 55–88
Spero A and Szafron D A 1978 *J. Math. Phys.* **19** 1536–41
Szafron D A 1977 *J. Math. Phys.* **18** 1673–7
Szafron D A and Collins C B 1979 *J. Math. Phys.* **20** to appear
Szafron D and Wainwright J 1977 *J. Math. Phys.* **18** 1668–72
Szekeres P 1975 *Commun. Math. Phys.* **41** 55–64
Wainwright J 1974 *Int. J. Theor. Phys.* **10** 39–58
Wainwright J, Ince W C W and Marshman B J 1979 *Gen. Rel. Grav.* **10** 259–71
York J W 1971 *Phys. Rev. Lett.* **26** 1656–8